SPIN, STATISTICS AND SPACE-TIME

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Abstract

A statistics may be regarded as a functor from individuals to composites. Each of the classical groups generates a unique natural quantum statistics. The A groups generate Fermi-Dirac, the C Bose-Einstein, and the B and D, the newer Schur-Wilczek statistics, where the individual is described in a quadratic space and the composite in its spinor space. S-W is intermediate between F-D and E-B in the sense that swaps, which are $+1$ for E-B statistics and -1 for F-D, have eigenvalues ± 1 or $\pm i$ for S-W. Of these statistics only the S-W is 2-valued and gives rise to spinors. Space-time points are therefore likely to be S-W, with spin arising from more fundamental swap. We argue that below the quark scale but far above the Planck scale, space-time, matter, measurement and the dynamical law are no longer distinct but fuse into one variable, the dynamic of the system; a localized refinement of the S matrix theory of Heisenberg. S-W statistics then implies a Clifford-algebraic language for physics, connecting the spinorial chessboard to the four-dimensionality of space-time.

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Dedication

This contribution carries forward the pleasurable discussions of quantum logic and quantum set theory that John Stachel and I had at Stevens Institute of Technology in the 1950's. It is a privilege to have had him as classmate and colleague, and I wish him many more productive years.

1 Beyond quantum set theory

In the intervening moments, a plausible quantum set theory was formulated and applied to space-time with some encouraging results [Finkelstein (1996)]. Here I point out that set theory is just one of three main lines of combinatorics, one for each of the main lines of classical groups.

Each line of classical groups defines its own combinatoric algebra in a way described below.

Set theory and its quantum correspondent, Fermi-Dirac statistics, come from the A line of unitary groups. It was over-optimistic to hope that quantum sets might be the appropriate combinatorics for the elementary parts of space-time, but we had no better statistics in the 1950's.

The C line of symplectic groups brings us the Bose-Einstein statistics. The composite with B-E statistics we call the *sib*. A classical sib would be a sequence of any number of terms, all the same.

B-E statistics was briefly considered for space-time points [Finkelstein, Saller and Tang (1998)], but does not account for the stability of space-time against collapse.

I leave aside the exceptional groups and their exceptional statistics.

Finally, the B and D lines of orthogonal groups lead to the Schur-Wilczek statistics, the main point of this note. Where F-D gives rise to Fermi algebra and E-B to Heisenberg algebra, S-W statistics generates a Clifford algebra.

F-D and B-E statistics associate with each individual two nilpotent operators representing creation and annihilation. S-W statistics associates with each element one Clifford unit representing a swap.

S-W is the only double-valued statistics of the classical-group statistics, and so the most natural of them for space-time points. And the last to be discovered [Wilczek (1997). S-W statistics should not be confused with the earlier anyon statistics also associated with Wilczek.]

Composites with S-W statistics will be called *squads*. There are many classical sets and some classical sibs but no classical squads.

2 Concept of the dynamic

If we wish to study the dynamical development of a system under maximal quantum resolution, so that the quantum structure of space-time reveals itself, we must improve on several approximations of the present quantum theory.

In the quantum theory of Heisenberg and Bohr, the dynamics is regarded as a completely describable and therefore classical object, described for example by a Hamiltonian operator or an S matrix summarizing an infinite number of quantum experiments, sandwiched between inlet and outlet channels, as in

$$D = \langle\omega| \otimes S \otimes |\alpha\rangle \tag{1}$$

One then contracts D to find the transition amplitude. This familiar assumption of quantum theory is inaccurate on several counts.

First, due to their quantum structure and their gravitational fields, our experiments are not infinitesimal in size and infinite in number, but necessarily finite in size and number. Therefore our knowledge of the dynamics is incomplete in principle. We hypothesize therefore that the dynamics too is a quantum variable, with no complete description like (1) but only maximal ones, combining by quantum superposition.

Second, the propagator S is not truly independent of $\langle\omega|$ and $|\alpha\rangle$, the inlet and outlet actions that surround it, as (1) assumes. At very least, our boundary actions have inevitable gravitational consequences in the space-time they bound. These effects are estimated in Finkelstein (1999). They lead to a breakdown of field theory at large times T as well as small times τ, with

$$\tau \approx \sqrt{T_{\mathrm{P}}T} \gg T_{\mathrm{P}},$$

about 15 orders of magnitude above the Planck time if field theory holds as far down in the energy scale as 1 μeV. This is resolvable by energies as low as $\sim$ 100 TeV. Therefore we are already immersed in breakdowns of the classical continuum that we have not yet learned to see. One of these, presumably, is the mass spectrum of the elementary particles, which might reflect the Brillouin zone structure of the vacuum hypercrystal.

Third, (1) assumes that the dynamics D is a function of the other operators of the system. But given so rich a variable as the dynamics, one probably needs no other. Now the dynamics is our sole independent quantum variable.

We celebrate this monistic hypothesis by singularizing "dynamics." All variables of the system are now functions of its *dynamic*, a quantum matter-space-time-dynamical-law unity-in-multiplicity.

This inverts the current practice, where the dynamics is supposed to be a function of all the other variables. Both in the quantum standard model and in general relativity, the conceptual apparatus of space-time, field operators, and action principle is all directed to describing the dynamics, though under less than maximal resolution, as a function of other system variables. All of these investment must now be recovered from the dynamic.

The variable dynamic may be regarded as generalizing, localizing, and activating the fixed S matrix of Heisenberg. S matrix theory assumes an underlying fundamental flat space-time, ignoring Einstein locality. The space-time in the experimental chamber, like any other dynamical variable, is known only through the processes that go on, and so can be determined from the dynamic and should not be postulated. We assume neither flatness nor even any definite dimension or signature under high resolution.

We consider the simplest possibility first: that the atomic elements of the system undergoing the dynamic have no parts. We therefore call them points, after Euclid, and expect that the usual classical space-time points are the correspondence limit of one or several of these quantum points.

The space-time metric tensor is a poor man's dynamics. As the Lagrangian of a massless point particle, it defines, and is operationally defined by, how test bodies move under gravity. A test body is just a smallish system with only gravitational interactions whose tides do not effect its orbit.

The theory of the metric can therefore give precious hints toward a theory of the dynamic. We consider the usual quantum assumption (1) as a quantum analogue of flat-space-time physics. It presupposes a fixed dynamics and arbitrary initial and final actions as special relativity presupposes a fixed metric ad arbitrary space-time content.

In classical thought the path description varies from experiment to experiment on the same system, while the action is fixed. In the quantum theory, however, the dynamics assigns a probability amplitude to each path, and is therefore identical with a superposition of path descriptions, a highly entangled description of the one q path. This is complementary to the specification of the end-points or for that matter of any point on the path. In the c theory the endpoint specification is *supplementary* to the dynamics description, and completes it. In the q theory the endpoint data is *complementary* to the dynamics description. The existing quantum theory describes its dynamics sharply, as a single entangled path, while the classical theory describes its dynamics only crisply, as a set of paths.

We have yet to understand exactly how the approximation of a fixed dynamics can work so well if indeed the dynamics is as highly variable as I expect. Presumably it corresponds to the fact that the flat space-time metric is a good approximation for much of physics, despite the variability of the space-time metric, and there are several ways under study in which this could come about.

We encapsulate next two basic quantum assumptions about the dynamic of the above discussion. D1 connects the theory of the dynamic with standard quantum theory, and D2 separates it.

D1. Superposition *The maximal descriptions of of the dynamic are the linear operators on a finite-dimensional input vector space of the system.*

For this to correspond to the usual

$$A = \langle\omega|S|\alpha\rangle, \tag{2}$$

a D must contain information about all three phases of the experiment, input, throughflow and outtake, no longer approximated as independent, with transition probability amplitudes given by the trace

$$A = \operatorname{Tr} D. \tag{3}$$

In usual quantum theory In S is a sesquilinear space (a linear space with a non-singular sesquilinear form, possibly indefinite). Gauge invariance requires that the gauge generators be Hermitian, and the gauge group structure that some of them be nilpotent. Only in an indefinite sesquilinear space can a nilpotent be Hermitian.

The quantum theories that rest on Hilbert space, with its definite sesquilinear form, are therefore not sufficiently relativistic for our purpose. Each experimenter E uses positive dimensions of the space to represent E's actions upon the system S. The transformations of E to other experimenters E′ are often far more numerous than transformations of S, as in the Gupta-Bleuler quantum electrodynamics; for E is necessarily more complex than S. Therefore there are "passive" actions with no "active" counterpart. To represent them, E requires the negative dimensions as well as the positive.

D2.1 Atomism *The system is a composite of a finite number of points.*

The mode of composition, the statistics, is crucial. It is specified by D2.2 below.

Usually one assumes a continuous time coordinate, allowing us to analyze the dynamic into finer transformations without ever reaching non-composite or atomic transformations. This assumption is unphysical and leads to infinities. D2.1 cuts this process off and requires a finite scale time $\tau > 0$. As Hartland Snyder pointed out long ago, this does not conflict with exact Lorentz invariance within the framework of quantum (or non-commutative) geometry.

The quantum system on which the global dynamic D acts we designate by S. The dynamic is described by a linear operator on the input vector space In S of the system, so it transforms as a pair of systems: $\mathrm{D} \sim \mathrm{SS}^{\dagger}$. To represent how we compose S of points P we write $\mathrm{S} = \mathcal{Q}\mathrm{P}$, where $\mathcal{Q}$ is a quantifier defining the statistics of the point, discussed further in the next section. Similarly the global dynamic $\mathrm{D} = \mathcal{Q}\mathrm{X}$ is composed of microscopic elementary dynamics or chronons X, which transform as pairs of points: $\mathrm{X} \sim \mathrm{PP}^{\dagger}$. We thus have the commutative square of natural mappings

$$\begin{array}{cccc} & & \mathrm{End}: & \\ & \mathrm{P} & \rightarrow & \mathrm{X} \\ \mathcal{Q}: & \downarrow & & \downarrow \\ & \mathrm{S} & \rightarrow & \mathrm{D} \end{array} \tag{4}$$

The horizontal arrows End lead from individuals to their dynamics. The vertical arrows $\mathcal{Q}$ lead from individuals to their composites.

We may iterate $\mathcal{Q}$ or End and extend this diagram downward or to the right as necessary. For example, considered as a quantum individual, the dynamic D has an input vector space In D of maximal descriptions, represented by the vertex D of the above diagram, and also an algebra of properties, variables and tranformations, the algebra End In D = End End In S, a double algebra defining D as a quantum groupon, extending the diagram to the right. The dynamic then the generic element or groupon of a q semigroup.[1]

The number N of P's in S reflects the space-time extent of the process we choose to study. We take N to be finite, leaving any limit $N \to \infty$ for last. In ordinary quantum experiments $N \gg 1$.

The S matrix of Heisenberg describes a global evolution of a system, from infinite past to infinite future. Since we seek an analysis into elementary processes, we suppose that the dynamic describes only one step in history. In the case of autonomous systems, the whole story is constructed by iteration.

In order to acccount for spin 1/2 we assume the space In P is a real quadratic space provided with a quadratic form $|\alpha\rangle \cdot |\alpha\rangle = \langle\alpha|\alpha\rangle$ of some as yet unspecified dimension $N = N_+ + N_-$ and signature $N_+ - N_-$, rather than a Hilbert space. Then its automorphism group is an orthogonal group $\mathrm{O}(N_+, N_-)$.

Since we have assumed that the point P has no internal structure, it has only its relations to other points, and all system variables can be expressed in terms of point permutations. This is the extreme opposite of the F-D and B-E statistics, where *no* system variables can be expressed in terms of particle permutations ($= \pm 1$). Any operator n whose N eigenvalues number the points in a composite and distinguish them from one another is a maximal set of commuting operators for the individual point by itself.

The one-point orthogonal operators $O : \mathrm{In\,P} \to \mathrm{In\,P}$ include the permutations of identical particles that usually figure in statistics, but are more general; so I will call them *permutors*.

3 Point statistics

For a composite of indistinguishable particles one usually defines statistics by a representation of the permutations of the individuals in the composite. This raises a semantic question: What quantum physical operations can be meant by such an exchange?

Sometimes one considers permutation as a homotopy. One imagines first gradually erecting potential walls around two regions to form boxes that contain the structures to be interchanged, then continuously interchanging the two boxes, and then lowering the potential walls, removing the boxes. This adiabatic process may result in a Berry phase change.

A homotopy, however, is not a true permutation of atomic parts, and may have little to do one. Nor does a homotopy theory of particles relieve the ultraviolet divergences of field theory.

The idea of the permutation group acting on a set of quanta arose from a classical preconception. First the composite was erroneously represented by a tensor product, and then this error was corrected by specifying a representation of the permutations of the parts, namely by scalars.

[1] In the sense of Finkelstein (1996). Our q semigroup is defined by a double algebra that need not be Hopfian. The Hopf property seems to be a classical vestige, appropriate when there is a classical space-time underlying the network.

We prefer not to found a correct theory on an erroneous hypothesis. We turn our attention from unfeasible permutations to feasible actions. The quantum theory must tell us what actions are possible.

By the *statistics* of a quantum we shall mean the quantification functor $\mathcal{Q}$, assumed to exist, from the elementary system s to the composite system $\mathrm{S} = \mathcal{Q}\mathrm{s}$. In the present work,for example, the elementary system is the point P and $\mathrm{S} = \mathcal{Q}\mathrm{P} = \mathrm{Sq}\,\mathrm{P}$. Historically the quantification functor $\mathcal{Q}$ has been called "second quantization" by physicists.

Our reformulation obviously encompasses the existing usage. For F-D statistics, the quantification functor is $\mathcal{Q}$ = Grass, which forms the Grassmann (exterior) algebra of the vector space on which it acts. For E-B, $\mathcal{Q}$ = Sym, which forms from any vector space the symmetric tensor algebra of that space. For M-B statistics, $\mathcal{Q}$ = Ten, which forms the tensor algebra over its vector space argument.

The functor $\mathcal{Q}$ by definition also defines a representation of the morphisms of In P by morphisms of the composite space In S.

We imbed permutations in the orthogonal group by applying them to the axes of a frame in the input vector space of the individual system. Then relative to any frame in In P, $\mathcal{Q}$ also defines a representation of the permutation group $\mathrm{S}_N \subset \mathrm{O}(N)$, thus subsuming the usual concept of statistics.

Now the classical concept of permutation group dissolves into the quantum concept of the automorphism group of the input vector space of the individual. Permutations have no distinguished role within the larger group of orthogonal transformations of the individual.

For any one frame, to be sure, the permutations of the frame vectors form a subgroup of the automorphism group. But this is not an invariant subgroup and has no invariant meaning.

Permutations are just the automorphisms of a classical discrete state space. The automorphisms of a quantum system are the non-singular isometries of its mode-vector space. In that sense the orthogonal transformations of the individual mode-vector space are the quantum analogues of the classical particle permutation operators. This is why we refer to these linear operators as permutors. Unlike permutations, however, the permutors form a continuous Lie group and a semisimple group. This is where the classical groups come in.

The representations of the permutation group used in the usual statistics are fully reducible, being commutative. The entire Hilbert space reduces to a direct sum of 1-dimensional rays. In the present non-commutative statistics we must instead demand an irreducible representation of the permutors, since all operations are expressed in terms of permutors. To represent the spin 2-valuedness correctly, we also required [Finkelstein and Gibbs (1993), Finkelstein (1996)] the representation to be 2-valued.

3.1 Classical-group statistics

Each of the classical groups G leads naturally to its own special statistics as follows. Recall that each classical group G can be identified with the automorphism group of a unique associated scalar-product module. Interpret this module as the input module In I of a variant individual quantum system I. The many-system quantification is defined by giving the operator algebra of the quantified I as a quotient of the tensor ring of In I modulo Γ-invariant quadratic commutation relations

$$u_m u_l c^{ml}{}_{kj} - f_{kj} = 0 \tag{5}$$

among the basis vectors u_m of In I. Here f is the G-invariant scalar-product form on In I and the commutation tensor c is a G-invariant bilinear form

$$\delta^m{}_k \delta^l{}_j \pm \delta^l{}_k \delta^m{}_j \tag{6}$$

in the basis vectors, with the sign fixed so that c has the symmetry of f.

It is easy to see that the A series, with its Hermitian symmetric scalar product, leads to Fermi-Dirac statistics, and the C series, with its antisymmetric inner product, to Bose-Einstein. Here is the whole pattern, omitting the exceptional groups:

Group G:	*A*	*B-D*	*C*
1-body vector space:	Hilbert	Quadratic	Symplectic
Statistics:	F-D	S-W	B-E
Composite:	Set	Squad	Sib
N-body ring:	Fermi	Clifford	Heisenberg
N-body module:	Grassmann	Spinor	Symmetric tensor

The groups G of the B and D series act on real quadratic spaces In I $= \mathbb{R}(N_+, N_-) := N_+\mathbb{R} \ominus N_-\mathbb{R}$ with dimension $N_+ + N_-$ and signature $N_+ - N_-$. They lead to a 2-valued irreducible representation of permutations, unlike the statistics of any of the presently known quanta.

The 2-valued ("fractional linear") irreducible representations of the permutation group were reported by Wiman (1898) and enumerated and described by Schur (1911). They have been investigated by Hamermesh (1962), Karpilovsky (1985), Stembridge (1989), Hoffman and Humphreys (1992), and many others. Finkelstein and Gibbs (1993) and Finkelstein (1996) used a 2-valued representation of the permutation group for space-time points without recognizing it as a variant statistics. Wilczek (1997) is the first to have recognized projective or 2-valued statistics as a statistics, proposing Schur's spinor representation of S_N for quasi-particles of the fractional quantum Hall effect. I therefore call the projective statistics of the $B - D$ groups the Schur-Wilczek or S-W statistics.

A composite with S-W statistics I call a *squad* and write S = Sq I.

Maxwell-Boltzmann statistics is the case $f \equiv 0$; A composite with M-B statistics is known as a *sequence* so we designate the M-B quantification functor by $\mathcal{Q}$ = Seq.

The many-body module shown for each group is the "square root" of the many-body operator ring above it in the sense that, schematically speaking, ring $\sim$ module $\otimes$ module†.

In the A and C lines the many-body module has an invariant grade. This makes these statistics appropriate for indistinguishable elements, which are counted by the grade. The $B - D$ lines have ungraded modules. We use them for distinguishable elements. The number of elements is the dimension of the one-body module.

There are classical sets and sibs as well as quantum but no classical squads. The squad incorporates superposition more deeply than the sequence, sib, or set. Calling it a sQUAd reminds us of its intrinsic QUAntum content.

In the early days of quantum physics, the 2-valued representations of the rotation group were overlooked for a time because they do not occur in a tensor product of spinless quanta. We overlooked the 2-valued representations of the permutation group and 2-valued statistics for a longer time for much the same reason: They do not arise in a tensor product of the vector spaces of individual quanta.

The S-W statistics permits a simpler proposal for the point than was possible before:

D2.2 Point statistics *The system is a squad of points.*

That is, actions on the system S are maximally represented by rays in a Clifford algebra $\mathrm{Cliff}(N_+, N_-)$ over a quadratic space $\Sigma = \mathrm{In}\,\mathrm{P} = \mathbb{R}(N_+\, N_-)$, instead of operators on a Hilbert space. We call an element of this Clifford algebra a *cliffor*.

3.2 Space-time dimension as order parameter

In a theory of space-time quanta, the dimension and signature of space-time are order parameters of the vacuum condensate, and one searches for some reason why the effective space-time of the classical continuum limit is a Maxwell-Boltzmann composite (sequence) of dimension 4 and signature 2 or -2, and why the effective quantum theory of systems in that space-time is complex rather than real.

S-W statistics is not inconsistent with a quantum condensation. Although usually we associate condensates with E-B statistics, we have never actually seen a true B-E condensate. Liquid He^4, for example, is a condensate of fermions. Electrons, protons and neutrons form quasibosonic octets, He atoms, and thereby condense into a superfluid. Likewise superconductivity is an F-D condensation that proceeds via Cooper pairs instead of Helium octets.

In S-W statistics, squads of eight (*octads*) and four (*tetrads*), are special, yhaks to the periodicities of the spinorial "clock" [Atiyah, Bott, Shapiro (1964)] and "chessboard" [Budinich, Trautman (1988)]. The *squad* of $4N$ points, for example, is algebraically isomorphic to the *sequence* of N tetrads.

This means that points with S-W statistics naturally form into tetrads with quasi-M-B statistics. This quantum M-B space includes a B-E subspace in which condensation can proceed as usual. Each of these tetrads with signature 3-1 or 4-0 has input spinors of four real components, essentially Majorana spinors. (Tetrads of signatures 2-2, 1-3 or 0-4 have input spinors with 2 quaternionic components or 8 real components.) In the present theory it is natural to relate this mathematical fact to the observed statistics and dimensionality of space-time in the classical continuum limit, where M-B statistics has implicitly ruled since Euclid.

The one theory to my knowledge in which the Higgs field is not an *ad hoc* add-on but a structural element arising from a deeper principle is one which a variable Clifford element $\eta(x)$ serving as $i\hbar$ in the quantum action principle [Finkelstein *et al.* (1962)] is discovered to be a natural Higgs field. In this η theory the Clifford algebra that appears is Cliff$(0, 2)$, the quaternions, and so there was enough Clifford materiel to construct SU_2 isospin but not SU_3 color, and we abandoned the Clifford road. S-W statistics gives us embarrassing riches of fundamental Clifford units, and so we take to the Clifford road once more.

3.3 Relativistic content, quantum form

Quantum kinematics has a simple basic syntax. It bi-uniquely corresponds each action on the system S and each tranformation of the experimenter with an endomorphism of the projective geometry of an input vector space In S. This endomorphism in turn is *projectively*, and therefore non-uniquely, represented by a linear operator on In S. This syntax applies equally well to an oscillator, an atom, a field or a crystal as long as it is isolated between our determinations of it. In application to a field theory the quantum algebraic syntax is necessarily non-local because its basic concepts and operators refer to entire space-like surfaces. We understand this to be a valid representation of the quantum experimenter, who is a global entity, not a local one.

Quantum kinematics focuses our attention on the operation semigroup and gives its possible structures. It gives us an open grammatical form that we have to fill with specific physical content by giving experimental meaning to the operators. It gives no hint at all what system to study.

General relativity, on the contrary, tells us exactly what to study: the causal relations among events, as revealed by signaling operations. These are local relations, defining the light-cone field of space-time, but they are not quantum. General relativity omits fine detail.

To integrate the quantum kinematics with the general relativistic dynamics requires us to give quantum form to a relativistic content.

The deepest quantum algebra we have today is that of the standard model, generated by gauge, leptoquark and Higgs fields dwelling on a frozen Minkowski space-time manifold without gravity. A separate classical theory based on general covariance and equivalence principles describes the gravitation of smoothed distributions of energy and momentum. This split is clearly a transitional phase. Intensive search for a simpler, deeper algebra goes on widely today.

The algebra we seek no longer represents a field theory. All field theories (including string and membrane theories) inherit a basic one-way coupling from space-time to field, evidence of a contraction in the Inonu-Wigner sense [Inonu and Wigner (1952); Marks *et al.* (1999)]. They also suffer from the measurement problems of the Bohr-Rosenfeld-DeWitt kind already mentioned, associated with horizon production. These genetic defects indicate that the present field theories are degenerations of a more atomistic quantum theory of the dynamic, with at least one new fundamental constant, a time τ, comparable to the time formed from the Higgs mass and greater than the Planck time T_P by many orders of magnitude.

We must expect discreteness only for time eigenvalues resulting from single measurements, not for expectation values like scattering lengths, which can be arbitrarily small.

The theory of the quantum dynamic must degenerate both to a q field theory and to general relativity, perhaps in two distinct appropriate singular limits. One simplification of q space-time theory compared to c is that in the limit $N \to \infty$ the q theory can be exactly invariant under the Poincaré group and the standard model groups even for finite τ. It would suffice if in the limit $\tau \to 0, N\tau \to \infty$ we recover classical space-time and general relativity; and in the limit $N \to \infty, \tau \sim 1$, quantum field theory.

3.4 Relativization of locality

The locality concept must be relativized by such a consolidation of quantum and relativity theory. The absolute locality principle so basic to Einstein is somewhat alien to quantum theory, though incorporated in our present hodge-podge relativistic quantum field physics. Sudarshan pointed this out to me years ago, but it conflicts so strongly with the locality principles of general relativity that I heed him only now.

The conflict is indicated at the most elementary level by the fact that from the algebraic point of view that is supposed to dominate quantum theory, there is no absolute difference between position and momentum. One may transform from position-fixing modes to momentum-fixing modes with a unitary transformation, an application of the superposition principle. But locality refers specifically to position, not momentum. For a local interaction to occur between objects, their positions in space must agree at some time, but they can be far apart in their momenta. Locality breaks the unitary symmetry of quantum algebra.

In the present context, to make an absolute distinction between position and momentum at the one-point level would break the orthogonal invariance of the mode space of the quantum point.

We infer that the chronon has an 8-dimensional space has an orthogonal symetry group and the familiar distinction between position and momentum arises from a condensation of many chronons into the usual space-time.

Therefore the dynamic has no absolute locality concept but only a relative locality, fixed by the dynamic D. While the condensation of a manifold out of a collection of points is quite special and requires $N \to \infty$, in that special case the system variables and their representation in the composite are quite numerous enough to define both position and momentum variables and distinguish between them.

Physics has progressed little in the direction of quantum space-time. Such important programs as quantum gravity, supergravity, grand unified theory, string theory, non-commutative geometry, and the standard model all still incorporate absolute concepts of locality and space-time, while our quantum relativity program to relativize them is still in a formative phase.

4 Transporting and transforming

Quantum theory and general relativity are both theories of composites, but they differ in how they how they connect their elements to form these composites. To reconcile these differences we first formulate them.

In general relativity the entities composed include the values of classical fields (like gravity) at different points of space-time. In order to agree with experience, and specifically with locality and the equivalence principle, when we compare field variables at different points, as in forming field gradients, we supplement them in practice with interpoint variables, connections defining transports of the fields along paths joining their points. The relativistic whole is more than the product of the original parts by these gauge transport variables, which now dominate modern particle physics and general relativity.

In elementary quantum mechanics, however, the parts are represented by input vectors ψ, $\phi, \ldots$ of the subsystems, and the whole is represented by a product $\psi \vee \phi \vee \ldots$ of its parts. The parts are not compared by continuous transport as in general relativity but, as different modes in the individual Hilbert space, by transposition in the usual concept of statistics, and by superposition in the present concept.

A statistics is called commutative (or Abelian, or central) if it represents all permutations by commuting operators, such as numbers. The B-E and F-D statistics are commutative, but not the M-B or S-W.

4.1 Connection variables

A system of indistinguishable particles has no connection variables. We can swap two subsystems in any of these products without further intersystem structure of the kind demanded by Einstein locality. The quantum algebra of the whole is merely a product of its parts.

To supply the necessary intersystemic variables, the standard model first assumes a classical fiber bundle theory providing all the intersystemic variables seen in nature, and then attempts to quantize this theory, using Bose statistics for the intersystemic quanta, Fermi for the leptoquark sources, and, implicitly, M-B for c space-time points. This hodge-podge has not worked for gravity beyond the weak-field approximation, and is clearly a temporary and provisional method for dealing with the other interactions.

The S-W statistics provides new intersystemic variables, the operators transforming its individuals into each other. The composite with this statistics is not a product of its parts. Besides the variables of the parts, It has new exchange variables, finite and infinitesimal; we refer to them as swaps and orthospins respectively.

S-W statistics introduces new variables to represent permutations just as new variables were introduced in the early days of quantum physics to represent orthogonal transformations in geometrical space, namely spins.

Here the basic new variable is the *swap* operator τ_{mn} that projectively represents the finite transformation

$$(mn) : \psi_m \mapsto \psi_n, \psi_n \mapsto \psi_n, \psi_k \mapsto \psi_k, k \neq m, n \tag{7}$$

a basis-dependent orthogonal transformation within the individual vector space V. *Spin* represents the infinitesimal orthogonal transformation

$$\Lambda_{mn} : \psi_m \mapsto \psi_n, \psi_n \mapsto -\psi_n, \psi_k \mapsto 0, k \neq m, n. \tag{8}$$

When necessary to distinguish this from a Lorentz-group spin operator, we call it *orthospin*. Both orthospin and swap are represented by operators on orthospinors.

Swap and orthospin are conceptually more primitive than (Lorentz) spin as permutation is more primitive than rotation, invoking no spatial concepts of length or angle but only quantum algebra. Yet much of classical Euclidean geometry in N dimensions can be formulated with swaps of N structureless quantum points. It seems conceivable that all physical actions and gauge transformations are ultimately composed of swaps of space-time points.

The vacuum is represented by a cliffor with a crystalline symmetry q group. Standard-model field-quanta are defects in the vacuum crystal structure, created and annihilated by swaps as patterns in a fabric are created and annihilated by transposing threads.

A seemingly many-valued statistics arises for quasi-particles in two-dimensional phenomenological theories like high-temperature superconductivity, even for systems of particles with 1-valued statistics, because there are many homotopically distinct paths by which two quasi-particles can be transposed. Each of these homotopy classes may result in a different quantum phase, depending on the enclosed gauge flux. We call such many-phased statistics "gauge statistics". When commutative, gauge statistics are projectively equivalent to 1-valued statistics.

For particles in three or more dimensions all transpositions are homotopic to each other and gauge statistics does not arise.

The fundamental 2-valuedness we treat here arises in any number of space-time dimensions, because it is not based on homotopy or even on the continuum, but on the fact that the permutation group on more than three objects is doubly covered by its representation group, the algebraic counterpart for finite groups of the universal covering group of a Lie group.

Braid statistics is an infinite-valued, infinite-dimensional representation of S_N. Since we restrict ourselves to finite-dimensional representations of $O_{N_+N_-}$, we do not consider braid statistics further.

4.2 The chronon and the Planck time

A crystal has many characteristic times with different physical meanings. So does the vacuum.

The space-time analogue of the crystal cell-size l is the fundamental time-scale τ of the space-time q cell (which may be converted to a length using lightspeed c). Several apparently independent thought experiments based on existing quantum field theory and general relativity indicate that the Planck time T_P is some 15 orders of magnitude smaller than τ: $\tau \gtrsim 10^{15}\, T_P$.

In our application of the coherent state method, some number $N_c \gg 1$ of quantum events cohere into one classical space-time event. Call N_c the *coherence number* of the vacuum. This is a vacuum analogue of the pure number $(\lambda_c/l)^D$ for a superconductor, where λ_c is the Ginzburg-Landau coherence length, l is the the cell size of the crystal, and $D = 2$ or 3 is the dimensionality of the superconductor.

Further, each mode that can not propagate in a crystal has a penetration depth $l_{\rm P}$. Similarly each mode to which the space-time q crystal is not transparent has a penetration number $N_{\rm P}$, the analogue of the penetration depth in units of the cell size. Large enough penetration numbers may manifest experimentally as masses or Compton wavelengths, for example of gauge fields.

A well-known elementary qualitative argument equates the Planck mass to a quantum black-hole mass. That is, the Planck length is not the cell size but is the penetration length of a high mode of the network.

One might expect length eigenvalues to have a discrete spectrum in a finite theory, with the cell size as the first eigenvalue. Then τ is the limiting precision of a meaningful non-zero eigenvalue of a continuum coordinate. A penetration length, however, is a mean value, not an eigenvalue. The Planck time is the limit of precision at which even an expectation value of a continuum coordinate can have meaning, solely due to gravitational effects. It is inevitable for mean values to occur that are much smaller than the first non-zero eigenvalue.

4.3 How F-D and B-E statistics can emerge from S-W

For each swap (lm) where ${\gamma_l}^2 = -1$ and ${\gamma_m}^2 = +1$, the operators

$$\begin{aligned} c = c(l,m) &:= \frac{\gamma_m - \gamma_l}{\sqrt{2}} \\ \bar{c} = \bar{c}(l,m) &:= \frac{\gamma_m + \gamma_l}{\sqrt{2}} \end{aligned} \tag{9}$$

obey the anticommutation relation of a fermionic creator c and annihilator $\bar{c}$. Then a set of such operators

$$c_\alpha := \frac{\gamma_{m_\alpha} - \gamma_{l_\alpha}}{\sqrt{2}}, \tag{10}$$

with no two having the same value of n_α or of m_α, taken with the $\bar{c}_\alpha$, obeys the fermionic anticommutation relations for the creators and annihilators of orthogonal modes $|\alpha\rangle$ of a fermion.

In turn, the fermionic creators and annihilators of an infinite sequence $|0\rangle, |1\rangle, |2\rangle, \ldots$ of such fermionic modes, each of which may be occupied or not, can be combined to make one boson creator and annihilator, in numerous ways.

S-W statistics describes a squad of 4 by a real spinor of 4 or 8 components, depending oh the signature of the squad. The representation used in Finkelstein and Gibbs (1993) and Finkelstein (1996) described four elements by a spinor of 24 components.

We maximally describe the space-time-matter-action dynamic D itself by a cliffor $D \in \text{Cliff}(N_+, N_-)$. A suitable D is to define both the action functionals of q field theory and gravity in appropriate degenerate limits.

Every sharp description of a c permutation is a collection of disjoint cycles. This is indeed a network, but a rather trivial one, a disconnected collection of disjoint loops of various sizes, without crossings or interaction vertices. The S-W statistics associates a Clifford extensor with every such permutation and its network. For physics, we need networks with crossings.

The operator algebra of the system of physics is now the possibly huge Clifford algebra End In S = Cliff $\mathbb{Q}$ In P. A q network is described by an element of this algebra. Operations on the network in turn, including space-time coordinates, infinitesimal translations, and statistical operators, are represented by linear operators *on* (not in) Cliff $\mathbb{Q}$ In P, that is, by second-order operators on Σ. Spin is presumably a concept of more limited validity than swap, requiring a local Minkowskian space-time for its meaning.

5 Does nature swap or spin?

We have given a connection between the classical groups and quantum statistics and used it to find a natural home on the B and D lines for the newest of the quantum statistics, the Schur-Wilczek. This is also the most plausible statistics for space-time points. For it is the only one that provides connection variables growing appropriately in number with the point population, the only one that gives rise to 2-valued spinor representations of rotations, the only one that deals with distinguishable elements, and the only one that leads to an M-B sequence of tetrads like the M-B assembly of the usual space-time points, with four dimensions and all.

It seems that the spinorial chessboard may be intimately connected with the fine-structure of space-time, as intimated by the logician C. L. Dodgson.

Because intrinsic spin is so much simpler than orbital angular momentum and the other variables of quantum field theory, space-time architects like Penrose (1971) have considered that spin might be more fundamental than space-time. But when the space-time is resolved into quantum elements and the classical Lorentz group loses meaning, it is not easy to give physical meaning to spin. I propose that spin derives from swap, and the 2-valued spin representation from a deeper 2-valued statistics.

Empirically, spin and statistics are correlated for quanta, with the operator W of 2π rotation equal to the operator X of two-quantum transposition. The combinatory structure we study now lies below the particle level. There the spin-statistics law is ignored even in present theories, in the sense that classical space-time points have no internal degrees of freedom and transform as spin-0 (invariant) entities, but have M-B statistics, not B-E, in being distinguishable. Earlier we proposed that the spin-statistics correlation arose from the fact that the processes W and X are homotopic [Finkelstein and Misner (1959)]. It seems now that its source is still deeper; that spin is correlated with statistics because fundamentally spin *is* statistics, and specifically is swap, a projective permutation operator.

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